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The Arm Fourier Theory

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Abstract

We give the developpment of functions in $\mathbb{C}[z] \oplus \mathbb{C}[z^{-1}]$ with a scalar product which involves an integral and a residue calculus. Then we give some examples of those developpments and find new 'representations' of the exponential and the logarithm function . We draw those representations and we see that there are similar to their original representations.

Introduction

The Arm Theory [1] gives the decomposition of functions in $\mathbb{C}[z]$. With the Arm shifted developpment, we can obtain the developpment of functions in the 'shifted' polynomial basis $z^m\mathbb{C}[z]$ with $m \in \mathbb{R}$. In fact, we use to work with functions in $\mathbb{C}[z]$, it is difficult for us to calculate with functions on $z^m\mathbb{C}[z]$. However we can imagine that functions on $\mathbb{C}[z] \oplus \mathbb{C}[z^{-1}]$ exist. Furthermore, to make an idea of what are those functions, we can imagine them as the representation of straight lines in the plan and functions in $\mathbb{C}[z]$ are the representation of half straight lines.

In effect, there is a problem if we use the scalar product based on the limit in zero and the infinty because, with the function in $\mathbb{C}[z] \oplus \mathbb{C}[z^{-1}]$, the left part ($\mathbb{C}[z]$) tend to infinity when we take the limit z tend to infinity. By contrast, the right part ($\mathbb{C}[z^{-1}]$) tend to infinity when z tend to zero. Therefore we need a new scalar product to obtain coefficients on $\mathbb{C}[z] \oplus \mathbb{C}[z^{-1}]$, this is why I had the idea to use the Fourier theory. To this end, if we transform each functions $f(z)$ into $f(e^{iz})$, we would obtain a 2π-periodic function which we can be developped on the Fourier basis $\mathbb{C}[e^{iz}] \oplus \mathbb{C}[e^{-iz}]$ to find its coefficients. Nevertheless there is functions which are not holomorph in some points (like $f(z) = \frac{1}{1-z}$ in $z = 1$), so we need to add a radius $r \in \mathbb{R}$ and a point z_0 for the complex number z which gives the Fourier transform of $f(re^{iz} + z_0)$. Here the generalization to each basis $u(z)$ is called the Arm Fourier decomposition of $f(z)$ on $\mathbb{C}[(u(z) - z_0)] \oplus \mathbb{C}[(u(z) - z_0)^{-1}]$ at z_0.

In other words, if we change of variable $z = z_0 + e^{iz}$, we can transform the Arm Fourier scalar product, which is an integral between 0 and 2π, into an integral on a circle of radius r and of center z_0 in the complex plane $\mathbb{C}$. Or rather if we use the modern complex analysis tools, we can apply the residue theorem which transforms the integration, which is not always easy to calculate because sometimes we do not know its primitive, into a quite simple residue calculus, which is a limit so more simple to calculate.

In the first section, we give the Arm Fourier developpment of $f(z) \in \mathbb{C}[(u(z) - z_0)] \oplus \mathbb{C}[(u(z) - z_0)^{-1}]$ at z_0 with r. Moreover, I want to stress that the first terms of this developpment is good nearby the neighborhood of the circle of center z_0 and radius r. In fact, the proof of the Arm Fourier formula takes the step I described above, i.e. transforming the function into a 2π-periodic function and finding its Fourier coefficient.

In the second section, I give the residue Arm Fourier formula (the expression of the Arm Fourier coefficient with residues) which makes the integral easier to be calculated. By the way, a residu is easier to be calculated because it is just a limit whereas calculate an integral is more difficult because we have to find a primitive which is not always easy. As a matter of fact, to calculate a developpment on $\mathbb{C}[z] \oplus \mathbb{C}[z^{-1}]$, we have to choose between the Arm Fourier developpment and the residue Arm Fourier developpment. To make this choice, we have to check if the function and its derivatives have non-infinity limits at the zeros of its denominator and in zero. As a consequence, functions which do not respect this condition (ex :ln) must be developped with the Arm Fourier developpment.

In the third section, we calculate some classical developpment of functions ($\frac{1}{1-z}$, $(1+z)^a$, $\ln z$) and we explicit all those calculus. As we said above, the begining developmment is correct if we look at the neighborhood of r because we consider real numbers. Over and above, we see that there is function (ln) which we can not calculate its residue so we have to take its Arm Fourier developpment. In addition we draw the first terms of the Arm Fourier developpment of $\ln z$ and we see that this developpment is not very good.

In the fourth section, we calculate the Arm Fourier developpment of $f(z) = z^{-\frac{1}{2}}e^z$ with r tends to infinity which gives us an other 'representation' of e^z when multipliying this developpment by $z^{\frac{1}{2}}$. In the same way, we calculate a new developpment of $\ln z$ nearby $r = 1$ on $z^{\frac{1}{2}}\mathbb{C}[z] \oplus z^{\frac{1}{2}}\mathbb{C}[z^{-1}]$. As a result, we draw those two new developpments and we see that the begining ot those developpments are good in the neighborhood of their own value of r.

1 The Arm Fourier Formula

If $f(z) \in \mathbb{C}[u(z)-z_0] \oplus \mathbb{C}[(u(z)-z_0)^{-1}]$ then $f(z)$ can be developped in the basis $..., (u(z)-z_0)^{-2}, (u(z)-z_0)^{-1}, 1, (u(z)-z_0), (u(z)-z_0)^2, ...$ and his developpment is given by the Arm Fourier formula.

Theorem 1. *If $f(z) \in \mathbb{C}[u(z)-z_0] \oplus \mathbb{C}[(u(z)-z_0)^{-1}]$, the Arm Fourier developpment of $f(z)$ is given by*

$$f(z) \quad = \quad \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} \left[\left(\frac{e^{-ix}}{r} \right)^k f\left(u^{-1}(z_0 + re^{ix}) \right) \right] \mathrm{d}x \right) (u(z)-z_0)^k \tag{1.1}$$

with $r \in \mathbb{R}$.

<u>**Proof :**</u>

Let $f(z) \in \mathbb{C}[u(z)-z_0] \oplus \mathbb{C}[(u(z)-z_0)^{-1}]$ be the function with the following decomposition on $\mathbb{C}[u(z)-z_0] \oplus \mathbb{C}[(u(z)-z_0)^{-1}]$:

$$f(z) = ... + \alpha_{-2}(u(z)-z_0)^{-2} + \alpha_{-1}(u(z)-z_0)^{-1} + \alpha_0 + \alpha_1(u(z)-z_0) + \alpha_2(u(z)-z_0)^2 + ... \tag{1.2}$$

where $..., \alpha_{-1}, \alpha_1, ... \in \mathbb{C}$.
Next the function

$$f\big(u^{-1}(z_0+re^{ix})\big) = ... + \alpha_{-2}(re^{ix})^{-2} + \alpha_{-1}(re^{ix})^{-1} + \alpha_0 + \alpha_1(re^{ix}) + \alpha_2(re^{ix})^2 + ... \tag{1.3}$$

can be developped with in the Fourier basis such that

$$f\big(u^{-1}(z_0+re^{ix})\big) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} f\big(u^{-1}(z_0+re^{ix})\big) \mathrm{d}x \right) e^{ikx} \tag{1.4}$$

Again we insert $z = u^{-1}(z_0 + re^{ix})$ which gives

$$f(z) \quad = \quad \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} f\big(u^{-1}(z_0+re^{ix})\big) \mathrm{d}x \right) \left(\frac{u(z)-z_0}{r} \right)^k$$

$$f(z) \quad = \quad \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} \left[\left(\frac{e^{-ix}}{r} \right)^k f\left(u^{-1}(z_0 + re^{ix}) \right) \right] \mathrm{d}x \right) (u(z)-z_0)^k \tag{1.5}$$

which give (1.1).

♦

Remark 1. *The Arm Fourier developpment is given with $r \in \mathbb{R}$, it is because there are function which can not be developped in $z = 1$(ex : $\frac{1}{1-z}$).*

Remark 2. *The passage (1.3) for $u(z) = z$ and $z_0 = 0$ is the well known compactification :*

$$\begin{array}{rcl} \mathbb{C} = \mathbb{R} \times \mathbb{R} & \longrightarrow & \mathbb{R} \times \mathcal{S}^1 \\ z & \longrightarrow & re^{ix} \end{array}$$

There is a problem with this compactification with the infinity point because it is identify to the point 0. It is because in all this theory when we encounter an infinity in a developpment, we formally put it as 0.

2 The Residue Arm Fourier Formula

The Arm Fourier developpment can be simplify with the residue theorem in the followingresidue Arm Fourier Formula

Theorem 2. *If* $f(z) \in \mathbb{C}[u(z)-z_0] \oplus \mathbb{C}[(u(z)-z_0)^{-1}]$*, the residue Arm Fourier developpment of* $f(z)$ *is given by*

$$f(z) \ = \ \sum_{k=-\infty}^{\infty} \left(\sum_{z_k \in Z_f \cap \mathrm{Int}(\gamma)} \mathrm{Res}\left(z^{-k-1} f(u^{-1}(z+z_0)) \ ; \ z_i \right) \right) \ (u(z)-z_0)^k \tag{2.6}$$

with $r \in \mathbb{R}$*,* γ *is the circle of center* 0 *and radius* r *and* $z_1, .., z_n$ *are the zero of the denominator of the function* $z^{-k-1} f(u^{-1}(z+z_0))$*.*

Proof :
From the Arm Fourier formula (1.1), we have

$$f(z) \ = \ \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} \left[\left(\frac{e^{-ix}}{r} \right)^k f\left(\, u^{-1}(z_0 + re^{ix}) \, \right) \right] \mathrm{d}x \right) \ (u(z)-z_0)^k \tag{2.7}$$

Next we change of variable such that $z = re^{ix}$ with $r \in \mathbb{R}$, and we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \left[\left(\frac{e^{-ix}}{r} \right)^k f\left(\, u^{-1}(z_0 + re^{ix}) \, \right) \right] \mathrm{d}x = \frac{-i}{2\pi} \int_\gamma z^{-k-1} f\left(\, u^{-1}(z+z_0) \, \right) \tag{2.8}$$

where $\gamma = \{ \forall z \in \mathbb{C} \text{suchthat} |z| = r \}$. Applying the residue theorem, we obtain

$$\frac{-i}{2\pi} \int_\gamma z^{-k-1} f\left(\, u^{-1}(z+z_0) \, \right) = \sum_{z_k \in Z_f \cap \mathrm{Int}(\gamma)} \mathrm{Res}\left(z^{-k-1} f(u^{-1}(z+z_0)) \ ; \ z_i \right) \tag{2.9}$$

which gives the Residue Arm Fourier formula (2.6).

♦

3 Examples

To find the Arm Fourier developpment of a function, I used an algorithm of the formula (1.1) which gives immediately the developpment. But if you want to calculate it yourself (without computer), the residue Arm Fourier formula (2.6) is more convenient. I caluclate here some usual developpment with the Arm Fourier theory

- The residue Arm Fourier developpment of $f(z)=\frac{1}{1-z}$ on the basis $u(z)=z$ with $r=\frac{1}{2}<1$ (or $u(z)=z^{-1}$ with $r=2>1$) at $z_0=0$ is given by :

$$\begin{aligned}
\frac{1}{1-z} &= \mathrm{Res}\Big(\frac{1}{z(1-z)};0\Big)+z\Big(\mathrm{Res}\Big(\frac{1}{z^2(1-z)};0\Big)\Big)+z^2\Big(\mathrm{Res}\Big(\frac{1}{z^3(1-z)};0\Big)\Big)+...\\
&= \Big(\lim_{z\to 0}\frac{1}{1-z}\Big)+z\Big(\lim_{z\to 0}\frac{\partial}{\partial z}\frac{1}{1-z}\Big)+z^2\Big(\lim_{z\to 0}\frac{1}{2!}\frac{\partial^2}{\partial z^2}\frac{1}{1-z}\Big)+...\\
&= 1+z\Big(\lim_{z\to 0}\frac{1}{(1-z)^2}\Big)+z^2\Big(\lim_{z\to 0}\frac{1}{2!}\frac{2}{(1-z)^3}\Big)+...\\
\frac{1}{1-z} &= \sum_{k=0}^{\infty} z^k
\end{aligned}\tag{3.10}$$

- The residue Arm Fourier developpment of $f(z)=\frac{1}{1-z}$ on the basis $u(z)=z$ with $r=2>1$ (or $u(z)=z^{-1}$ with $r=\frac{1}{2}<1$) at $z_0=0$ is given by :

$$\begin{aligned}
\frac{1}{1-z} &= \Big(\mathrm{Res}\Big(\frac{1}{z(1-z)};0\Big)+\mathrm{Res}\Big(\frac{1}{z(1-z)};1\Big)\Big)+z^{-1}\Big(\mathrm{Res}\Big(\frac{1}{(1-z)};1\Big)\Big)\\
&\quad +z^{-2}\Big(\mathrm{Res}\Big(\frac{z}{(1-z)};1\Big)\Big)+...\\
&= \Big(\lim_{z\to 1}\frac{z-1}{z(1-z)}+\lim_{z\to 0}\frac{z}{z(1-z)}\Big)+z^{-1}\Big(\lim_{z\to 1}\frac{z-1}{1-z}\Big)+z^{-2}\Big(\lim_{z\to 1}\frac{(z-1)z}{1-z}\Big)+...\\
\frac{1}{1-z} &= -\sum_{k=1}^{\infty} z^{-k}
\end{aligned}\tag{3.11}$$

- The residue Arm Fourier developpment of $f(z)=(1+z)^a$ on the basis $u(z)=z$ with $r=\frac{1}{2}<1$ (or $u(z)=z^{-1}$ with $r=2>1$) at $z_0=0$ is given by :

$$\begin{aligned}
(1+z)^a &= \mathrm{Res}\Big(\frac{(1+z)^a}{z};0\Big)+z\Big(\mathrm{Res}\Big(\frac{(1+z)^a}{z^2};0\Big)\Big)+z^2\Big(\mathrm{Res}\Big(\frac{(1+z)^a}{z^3};0\Big)\Big)+...\\
&= \Big(\lim_{z\to 0}(1+z)^a\Big)+z\Big(\lim_{z\to 0}\frac{\partial}{\partial z}(1+z)^a\Big)+z^2\Big(\lim_{z\to 0}\frac{1}{2!}\frac{\partial^2}{\partial z^2}(1+z)^a\Big)+...\\
&= 1+z\Big(\lim_{z\to 0}a(1+z)^{a-1}\Big)+z^2\Big(\lim_{z\to 0}\frac{1}{2!}a(a-1)(1+z)^{a-2}\Big)+...\\
(1+z)^a &= \sum_{k=0}^{\infty}\binom{a}{k} z^k
\end{aligned}\tag{3.12}$$

- The residue Arm Fourier developpment of $f(z)=(1+z^{-1})^a$ on the basis $u(z)=z^{-1}$ with

$r = \frac{1}{2} < 1$ (or $u(z) = z$ with $r = 2 > 1$) at $z_0 = 0$ is given by :

$$\begin{aligned}
(1+z^{-1})^a &= \left(\mathrm{Res}\left(\frac{(1+z)^a}{z};0\right)\right) + z^{-1}\left(\mathrm{Res}\left(\frac{(1+z)^a}{z^2};0\right)\right) \\
&\quad + z^{-2}\left(\mathrm{Res}\left(\frac{(1+z^{-1})^a}{z^3};0\right)\right) + ... \\
&= \left(\lim_{z\to 0}(z+1)^a\right) + z^{-1}\left(\lim_{z\to 0}\frac{\partial}{\partial z}(z+1)^a\right) + z^{-2}\left(\lim_{z\to 0}\frac{1}{2!}\frac{\partial^2}{\partial z^2}(z+1)^a\right) + ... \\
(1+z^{-1})^a &= \sum_{k=1}^{\infty} \binom{a}{k} \; z^{-k} \\
(1+z)^a &= \sum_{k=1}^{\infty} \binom{a}{k} \; z^{a-k} \qquad (3.13)
\end{aligned}$$

4 Functions on $z^a\mathbb{C}[z^{-1}] \oplus z^a\mathbb{C}[z]$

The Arm Fourier developpment of $f(z) = z^{-\frac{1}{2}}e^z$ on the basis $u(z) = z$ with $r \to \infty$ at $z_0 = 0$ is given by :

$$z^{-\frac{1}{2}}e^z = \frac{2}{\sqrt{\pi}} \sum_{k=-\infty}^{\infty} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(k+\frac{3}{2})} \right) z^k \tag{4.14}$$

with $c = \frac{2}{\sqrt{\pi}}$ which gives

$$e^z = \frac{2}{\sqrt{\pi}} \sum_{k=-\infty}^{\infty} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(k+\frac{3}{2})} \right) z^{k+\frac{1}{2}} \tag{4.15}$$

We can check that

$$\frac{\partial}{\partial z} \sum_{k=-\infty}^{\infty} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(k+\frac{3}{2})} \right) z^{k+\frac{1}{2}} = \sum_{k=-\infty}^{\infty} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(k+\frac{3}{2})} \right) z^{k+\frac{1}{2}} \tag{4.16}$$

We can see the first terms of the exponential and draw it

$$e^z = \frac{2}{\sqrt{\pi}} \left(... + \frac{3}{8}z^{-\frac{5}{2}} - \frac{1}{4}z^{-\frac{3}{2}} + \frac{1}{2}z^{-\frac{1}{2}} + z^{\frac{1}{2}} + \frac{2}{3}z^{\frac{3}{2}} + \frac{4}{15}z^{\frac{5}{2}} + \frac{8}{105}z^{\frac{7}{2}} + ... \right) \tag{4.17}$$

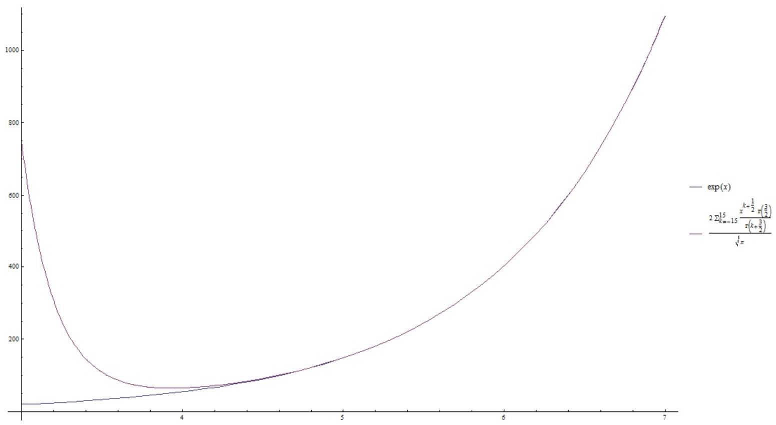

FIGURE 1 – The exponential on $z^{\frac{1}{2}}\mathbb{C}[z] \oplus z^{\frac{1}{2}}\mathbb{C}[z^{-1}]$

Doing the same thing with remark 2 we fing that the Arm Fourier developpment of $f(z) = z^{-\frac{1}{2}}e^{-z}$ on the basis $u(z) = z$ with $r \to \infty$ at $z_0 = 0$ is given by :

$$z^{-\frac{1}{2}}e^{-z} = \sum_{k=-\infty}^{\infty} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(k+\frac{3}{2})} \right) (-z)^k \tag{4.18}$$

which gives

$$e^{-z} = \sum_{k=-\infty}^{\infty} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(k+\frac{3}{2})} \right) (-1)^k z^{k+\frac{1}{2}} \tag{4.19}$$

Combining (4.15) and (4.19), we obtain the 2-exponential (which is nothing else that the hyperbolic cosinus see [2])

$$\cosh(z) \;=\; \sum_{k=-\infty}^{\infty} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(2k+\frac{3}{2})} \right) z^{2k+\frac{1}{2}} \tag{4.20}$$

and its derivate

$$\sinh(z) \;=\; \sum_{k=-\infty}^{\infty} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(2k+\frac{1}{2})} \right) z^{2k-\frac{1}{2}} \tag{4.21}$$

Of course we can check that

$$\frac{\partial^2}{\partial z^2} \sum_{k=-\infty}^{\infty} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(2k+\frac{1}{2})} \right) z^{2k-\frac{1}{2}} = \sum_{k=-\infty}^{\infty} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(2k+\frac{1}{2})} \right) z^{2k-\frac{1}{2}} \tag{4.22}$$

and

$$\frac{\partial^2}{\partial z^2} \sum_{k=-\infty}^{\infty} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(2k+\frac{3}{2})} \right) z^{2k+\frac{1}{2}} = \sum_{k=-\infty}^{\infty} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(2k+\frac{3}{2})} \right) z^{2k+\frac{1}{2}} \tag{4.23}$$

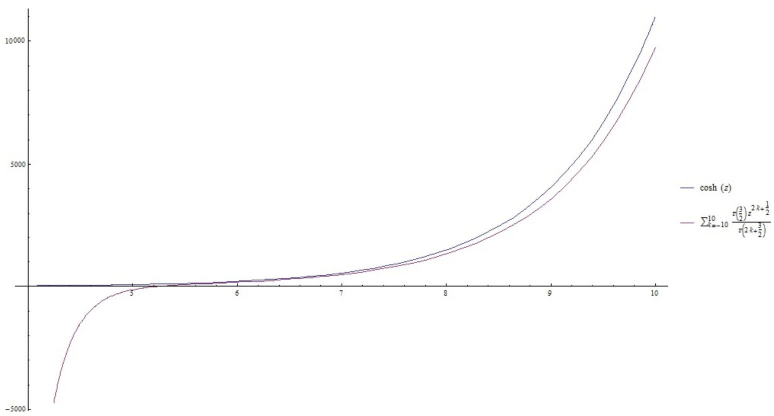

FIGURE 2 – The exponential on $z^{\frac{1}{2}}\mathbb{C}[z] \oplus z^{\frac{1}{2}}\mathbb{C}[z^{-1}]$

Now we consider the Arm Fourier developpment of $f(z) = z^{-\frac{1}{2}} \ln(z)$ on $u(z) = z$ at $z_0 = 0$ for $r = 1$:

$$\begin{aligned}
z^{-\frac{1}{2}} \ln(z) &= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} \; e^{-\frac{ix}{2}} \; \ln(e^{ix}) \, \mathrm{d}x \right) z^k \\
&= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} \; e^{-\frac{ix}{2}} \; \ln(e^{ix}) \, \mathrm{d}x \right) z^k \\
&= \sum_{k=-\infty}^{\infty} \left(\frac{i}{2\pi} \int_{-\pi}^{\pi} x \, e^{-i(k+\frac{1}{2})x} \; \mathrm{d}x \right) z^k \\
&= \frac{i}{2\pi} \sum_{k=-\infty}^{\infty} \left(\left[\frac{x e^{-i(k+\frac{1}{2})x}}{-i(k+\frac{1}{2})} \right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{e^{-i(k+\frac{1}{2})x}}{i(k+\frac{1}{2})} \, \mathrm{d}x \right) z^k \\
&= \frac{i}{2\pi} \sum_{k=-\infty}^{\infty} \left(\left[\frac{e^{-i(k+\frac{1}{2})x}}{(k+\frac{1}{2})^2} \right]_{-\pi}^{\pi} \right) z^k \\
&= \frac{i}{2\pi} \sum_{k=-\infty}^{\infty} \left(\frac{(-2i)(-1)^k}{(k+\frac{1}{2})^2} \right) z^k \\
z^{-\frac{1}{2}} \ln(z) &= \frac{4}{\pi} \sum_{k=-\infty}^{\infty} \left(\frac{(-1)^k}{(2k+1)^2} \right) z^k \qquad (4.24)
\end{aligned}$$

which can be expressed as

$$\ln(z) = \frac{4}{\pi} \sum_{k=-\infty}^{\infty} \left(\frac{(-1)^k}{(2k+1)^2} \right) z^{k+\frac{1}{2}} \qquad (4.25)$$

for $z \neq 0$ because $c = \frac{4}{\pi}$.

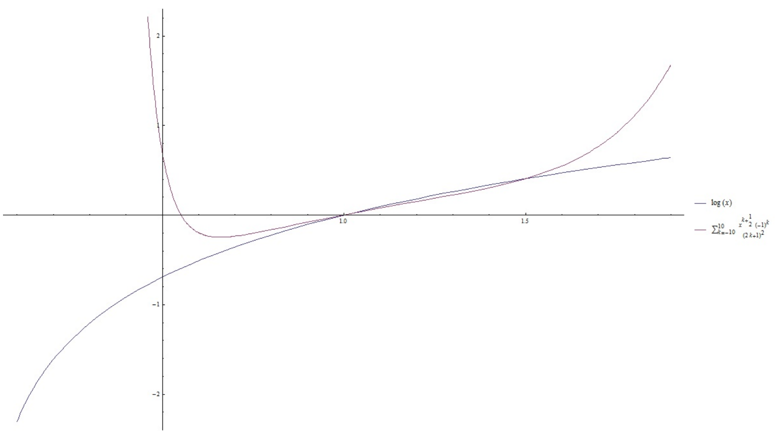

Figure 3 – The logarithm on $z^{\frac{1}{2}}\mathbb{C}[z] \oplus z^{\frac{1}{2}}\mathbb{C}[z^{-1}]$

From (4.25) we obtain

$$z^{-1} = \frac{1}{2} \sum_{k=-\infty}^{\infty} \left(\frac{(-1)^k}{(2k+1)} \right) z^{k+\frac{1}{2}} \tag{4.26}$$

Discussion

First, I introduced r in the Arm Fourier formula (1.1) to let the functions like $\frac{1}{1-z}$ to have their classical developpments because those functions can not be developped for variable of module 1. As a consequence, we needed to add a module to the function we wanted to deveopped on the Fourier basis and then divided by this module each component of the Fourier basis.

Next, when drawing the first terms of the developpment of functions, I saw that the developpment is faithful to the original function only in the neighborhood of r. In fact, the figure 1 represents the developpment of $\ln z$ on $\mathbb{C}[z] \oplus \mathbb{C}[z^{-1}]$ and we see that this developpment is not good. When derivating this expression, we obtain

$$z^{-1} = ... - z^{-3} + z^{-2} + 1 - z + z^2 + ... \tag{4.27}$$

which gives

$$0 = ... - z^{-3} + z^{-2} - z^{-1} + 1 - z + z^2 + ... \tag{4.28}$$

or

$$0 = - \lim_{z \to 1^+} \frac{1}{1+z} + \lim_{z \to 1^-} \frac{1}{1+z} \tag{4.29}$$

which is true but the each developpment are good only in their respective part of 1 and not in the other part.

Références

[1] Arm B. N., The Arm Theory

[2] Arm B. N., The p-Arm Theory